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Hydromagnetic edge waves and instability in reduction cells

By S. J. S. MORRIS[1] AND P. A. DAVIDSON[2]

[1]Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA

[2]Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, UK

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To clarify the function of gravity in the shallow-water theory of the interfacial instability in aluminium reduction cells, we analyse the existing long-wave theory of the instability in the limit of vanishing gravitational acceleration g. The flow then has an inner and outer structure, with gravity remaining essential within thin layers coating the cell walls. In those thin wall layers, the growing disturbance takes the form of a trapped magnetogravity wave propagating horizontally on the internal interface, and the growth rate σ is determined by the coupling of that edge wave to the large-scale flow in the core of the cell; that coupling is expressed as an oblique-derivative problem for the core flow. Although σ is asymptotically independent of g, gravity is essential to the long-wave instability because a correlation imposed by the magneto-gravity waves is essential for the disturbance to extract power from the mean state.

1. Introduction

Aluminium is produced by passing an electric current through a solution of alumina in molten cryolite. As the oxide is reduced, metal droplets form and sink to create an underlying dense liquid layer. To reduce electrical resistance, the overlying layer of electrolytic solution should be as thin as possible, but very thin layers are unstable to long waves on the internal interface. If allowed to grow, those waves short-circuit the cell. In an existing model of the instability, a liquid layer of low conductivity floats on a second, highly conductive liquid layer; both layers are of finite, constant thickness. A uniform vertical electric current J_o per unit area is passed across this layered system, within which there is also a uniform vertical magnetic field B_o caused by currents outside the cell. Hydrostatic equilibrium is possible for that configuration, but is unstable to long waves on the internal interface if the product J_oB_o exceeds a critical value (Bojarevics & Romerio 1994; Sneyd & Wang 1994; Davidson & Lindsay 1998).

Because the mechanism for the hydromagnetic instability is unclear, Davidson & Lindsay formulate a compound pendulum model. Like the reduction cell, their pendulum has an equilibrium state that loses stability if the product J_oB_o exceeds a critical value; gravity plays no part in causing that instability, but merely influences the critical value of J_oB_o. To connect the pendulum model to the hydromagnetic instability, the authors prove that the total momentum of the liquid aluminium satisfies the same ordinary differential equation (ODE) as the generalized coordinates of the pendulum; they infer that gravity plays no part in causing the hydromagnetic instability.

By contrast, we prove that gravity is central to the hydromagnetic instability described by the existing long-wave theory. In § 2, we state the equations governing that theory. In those equations, g multiplies the highest derivative, suggesting that analysis of the limit $g \to 0$ might give insight into the instability. In § 3, before analysing that problem, we give a new exact solution to the disturbance equations. It describes a trapped wave propagating on the aluminium–electrolyte interface bounded by a single plane wall; far from the wall, the interface displacement and current perturbation vanish. By writing a balance for an energy-like functional, we prove that the wave grows because power is fed to it from the background current via the coupling of displacement and perturbation current at the wall, and we conclude that gravity is essential to the instability because the edge wave determines the phase relation needed for power to be fed to the disturbance.

In § 4, we analyse the singular limit $g \to 0$ to prove that the conclusions drawn from the edge-wave solution also apply for a cell of arbitrary planform. In this limit, gravity affects the flow directly only in thin boundary layers coating the walls; the flow there is independent of cell planform, and so is a special case of our edge-wave solution. The growth rate of the instability is determined physically by the coupling of that boundary layer flow to the large-scale (core) flow in the rest of the cell. We show that the growth rate is asymptotically independent of g because the jump condition can be expressed purely in terms of core variables; however, gravity is essential to the instability because it establishes the flow structure from which that jump condition follows.

Because our analysis in § 4 also predicts that the growth rate is asymptotically independent of cell planform, we verify that result in § 5 using the instability in a channel. In that case there is also a critical value g_c of gravity, and we verify that the power supplied at the wall is positive if the base state is unstable, but vanishes otherwise.

In § 6, we relate our analysis to that of Davidson & Lindsay by proving that the velocity in the core of the cell satisfies the same ODE as the generalized coordinates in their pendulum model; but we also explain why it does not follow from their analogy that gravity is unimportant to the mechanism of the hydromagnetic instability.

2. Problem statement

Figure 1 shows the geometry of the reduction cell. The walls are vertical and all cell boundaries are rigid; only the fluid–fluid interface is free. The z-axis is vertical with origin at the equilibrium location of that interface; $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ are unit vectors in the coordinate directions. The cell planform occupies area $\mathscr{A}$. The inward unit normal to the curve $\mathscr{C}$ bounding $\mathscr{A}$ is $\hat{\boldsymbol{n}}$, the tangent $\hat{\boldsymbol{s}} = \hat{\boldsymbol{n}} \times \hat{\boldsymbol{z}}$; also s and n denote distance in the corresponding directions. A dot on a variable indicates a partial derivative in time t. In general, subscripts denote partial derivatives in spatial coordinates; the exception is that subscripts e and a refer to properties of the electrolyte and aluminium layers. The current $\boldsymbol{J} = \boldsymbol{j}_{a,e} + \mathscr{J}\hat{\boldsymbol{z}}$, and the velocities are $\boldsymbol{v}_{a,e} + w_{a,e}\hat{\boldsymbol{z}}$, so that $\boldsymbol{j}$ and $\boldsymbol{v}$ are horizontal vectors. The electrolyte has density ρ, and the metal, $\rho + \Delta\rho$. Unlike previous writers on this subject, we use the Boussinesq approximation because $\Delta\rho \ll \rho$ in practice. The unperturbed depths d_e and d_a are constant; the total depth $d = d_e + d_a$. The amplitude of the interface deflection is a; the scale L is half the

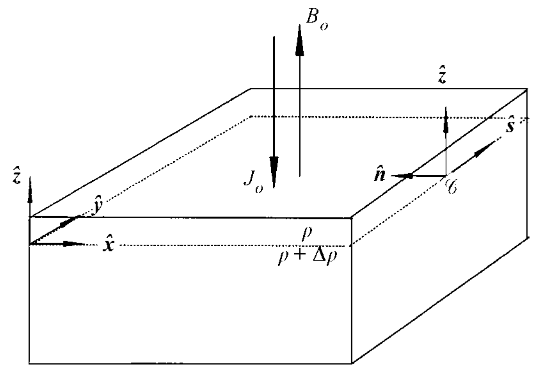

FIGURE 1. Definition sketch.

minimum horizontal dimension of the cell. The (Boussinesq) speed of a long internal gravity wave $C = \sqrt{\Delta\rho\, g d_a d_e/(\rho d)}$.

In the hydrostatic base state, the interface is horizontal, the current $\bar{\boldsymbol{J}} = -J_o\hat{\boldsymbol{z}}$, magnetic field $\bar{\boldsymbol{B}} = B_o\hat{\boldsymbol{z}} - \mu J_o x\,\hat{\boldsymbol{y}}$, and pressure gradient $\nabla\bar{p} = -(\rho' g\hat{\boldsymbol{z}} + \mu J_o^2\, x\,\hat{\boldsymbol{x}})$; here ρ' is ρ or $\rho + \Delta\rho$, for $z > 0$ or $z < 0$ respectively, and μ is the permeability of empty space. The horizontal field $-\mu J_o x\,\hat{\boldsymbol{y}}$ is induced by the uniform current $-J_o\hat{\boldsymbol{z}}$ within the cell, and the uniform vertical field $B_o\hat{\boldsymbol{z}}$ by currents external to the cell. In the base state, the liquids are stagnant because the body force is irrotational when the interface is horizontal.

In the perturbed state, the interface displacement is $\eta(x, y, t)$, and the pressure within the electrolyte extrapolated to $z = 0$ is $P(x, y, t)$. Infinitesimal long-wave disturbances are governed by the following equations:

$$\rho\dot{\boldsymbol{v}}_e = -\nabla P, \quad \rho\dot{\boldsymbol{v}}_a = -\nabla(P + \Delta\rho\, g\eta) + \boldsymbol{j}_a \times \hat{\boldsymbol{z}} B_o, \tag{1a, b}$$

$$d_e\nabla\cdot\boldsymbol{v}_e - \dot{\eta} = 0, \quad d_a\nabla\cdot\boldsymbol{v}_a + \dot{\eta} = 0, \tag{1c, d}$$

$$\boldsymbol{j}_a = \nabla\chi, \quad \nabla^2\chi = \eta J_o/(d_a d_e); \tag{1e, f}$$

$$\text{on } \mathscr{C}, \quad \hat{\boldsymbol{n}}\cdot\boldsymbol{v}_{a,e} = 0 = \hat{\boldsymbol{n}}\cdot\boldsymbol{j}_a. \tag{1g, h}$$

These are six equations for the six unknowns $\boldsymbol{v}_e$, $\boldsymbol{v}_a$, P, η, $\boldsymbol{j}_a$ and χ ; these variables depend only on x, y and t. Problem (1) is derived by Bojarevics & Romerio (1994, equations 3.33, 3.38, 3.39), and by Davidson & Lindsay (1998); our equations differ from those of the latter authors only because we make the Boussinesq approximation.

We see that (1) consists of the shallow-water equations (1a)–(1d) for a layered system, supplemented by a simplified form of the Lorentz force. That simplified form is appropriate because the conductivities of the electrolyte, the carbon electrodes forming the roof and floor of the cell, and the aluminium satisfy $\sigma_e \ll \sigma_c \ll \sigma_a$. Consequently, the disturbance current in the highly conductive aluminium sees the carbon floor of the cell as insulating, and so runs along the aluminium, remaining nearly horizontal. Conversely, the electrolyte is sandwiched between the carbon roof of the cell, and the highly conducting metal; the disturbance current follows the shortest path possible between the two good conductors, and so is approximately vertical. For long waves, it follows after some simple scaling analysis that within the electrolyte, the disturbance Lorentz force is negligible, whereas within the metal, the disturbance Lorentz force is $\boldsymbol{j}_a \times \hat{\boldsymbol{z}} B_o$. Further detail can be found in the literature.

The system is closed by (1e) and (1f), which together provide an equation for the current $\boldsymbol{j}_a$ in the aluminium. These equations follow from the conditions $\nabla \times \boldsymbol{J} = 0$ and

$\nabla \cdot \boldsymbol{J} = 0$ on the total current $\boldsymbol{J}$. The first of these expressions is the induction equation, simplified for quasi-steady fields at low magnetic Reynolds number, and (1e) follows by applying its vertical component to the aluminium. The second condition holds because $\nabla \times \boldsymbol{B} = \mu \boldsymbol{J}$ in the magnetohydrodynamic approximation. The continuity equation (1f) follows in the long-wave limit by analysing this pair of equations for $\boldsymbol{J}$ for both the electrolyte and metal, using the ordering of the electrical conductivities given above; (1f) states that regions where the electrolyte is locally thinned, so that $\eta > 0$, act as sinks of $\boldsymbol{j}_a$ because the thinner electrolyte layer allows more current to flow downwards into the metal.

We digress to discuss the conditions under which our analysis of the shallow-water theory (1) is meaningful. Edge waves are central to our argument, and (1) admits such waves because the simplified form of Lorentz force in (1b) is solenoidal by (1e). To learn precisely when that approximation holds, we note that because $\nabla \times \boldsymbol{B} = \mu \boldsymbol{J}$ and $\nabla \times \boldsymbol{J} = 0$, without further approximation $\nabla \cdot (\boldsymbol{J} \times \boldsymbol{B}) = -\mu J^2$. The perturbation to this divergence is $\sim \mu J_o \chi d/L^2$, because $\nabla \cdot \boldsymbol{J} = 0$ requires the perturbation in vertical current to be $\sim j_a d/L$. But by (1f), $\chi \sim a J_o L^2/d^2$. Consequently, the divergence of the perturbation Lorentz force $\sim J_o B_o L/(\Delta\rho\, g d)$ in comparison with the divergence $\Delta\rho\, g \nabla^2 \eta$ of the body force in (1b). The relation $B_o \sim \mu J_o L$ has been used. The error made by taking the Lorentz force as solenoidal is therefore negligible if

$$d/L \ll \mathscr{G}, \quad \mathscr{G} = \Delta\rho\, g d_a d_e/(J_o B_o L^2). \tag{2a,b}$$

In the derivation of (1), the limit $d/L \to 0$ has already been taken, and it follows from (2a) that it is legitimate to analyse the resulting model (1) in the limit $\mathscr{G} \to 0$; but (2a) also implies that for small $\mathscr{G}$, there also exist long-wave disturbances not described by (1). It would be interesting to see the case $g = 0$ analysed without using shallow-water theory. We return to our analysis of (1).

We non-dimensionalize (1) using scales that are independent of gravity; specifically, the time scale on which the Lorentz force balances inertia is

$$T = \sqrt{\rho d/(J_o B_o)}, \tag{3}$$

and we introduce dimensionless variables (without asterisks):

$$(x, y)_* = L(x, y), \quad t_* = T\, t, \quad \boldsymbol{v}_* = \frac{aL}{d_e T}\boldsymbol{v}, \quad \eta_* = a\,\eta, \quad \chi_* = a\frac{J_o L^2}{d_a d_e}\chi. \tag{4a–e}$$

We have chosen the scale for χ by balancing terms in (1f).

After non-dimensionalizing (1), we take the divergence of the difference of (1a) and (1b), then use (1c) and (1d) to eliminate $\boldsymbol{v}_e$ and $\boldsymbol{v}_a$. This shows that the displacement $\eta(x, y, t)$ and potential $\chi(x, y, t)$ satisfy

$$\ddot{\eta} = \mathscr{G}\nabla^2\eta, \quad \nabla^2\chi = \eta \text{ within } \mathscr{A}; \tag{5a,b}$$

$$\text{on } \mathscr{C}, \quad \chi_n = 0, \quad \mathscr{G}\eta_n = -\chi_s, \tag{5c,d}$$

where (5d) is derived by forming the difference of the normal component of the momentum equations, then using (1g) to eliminate $\hat{\boldsymbol{n}} \cdot \boldsymbol{v}_e$ and $\hat{\boldsymbol{n}} \cdot \boldsymbol{v}_a$. The instability is thus governed by the wave equation for η, and the Poisson equation for χ. These equations are coupled only at the boundary because the disturbance Lorentz force is solenoidal in the long-wave approximation, as discussed above.

To clarify the instability mechanism, we form a balance equation for the quantity $\mathscr{E} = \frac{1}{2}\int\{\dot{\eta}^2 + \mathscr{G}(\nabla\eta)^2\}\,\mathrm{d}\mathscr{A}$. (If (5$a$) described the displacement η of a membrane, $\mathscr{E}$ would be the mechanical energy, and for brevity, we call $\mathscr{E}$ the disturbance energy.) By multiplying (5a) by $\dot{\eta}$, integrating the result over $\mathscr{A}$, and using the divergence theorem,

$$\dot{\mathscr{E}} = -\mathscr{G} \oint_{\mathscr{C}} \dot{\eta}\, \eta_n \, \mathrm{d}s = \oint_{\mathscr{C}} \dot{\eta}\, \chi_s \, \mathrm{d}s, \qquad (6a, b)$$

where we have used (5d). Identity (6a) implies that the limit $\mathscr{G} \to 0$ is indeed singular; for if we were to ignore the warning given by (2a), and set $\mathscr{G} = 0$, $\mathscr{E}$ would be independent of time. It follows that if there is to be an instability for $\mathscr{G} \to 0$, gravity must remain important near the wall where the boundary integral in (6a) is evaluated. Further, from (6b) we infer that gravity will remain important by causing the phase relationship between $\dot{\eta}$ and χ_s to be such that the right-hand side of (6b) is positive.

We shall need the velocities later to interpret the flow; their time-derivatives are

$$\dot{\boldsymbol{v}}_e = \nabla \ddot{\chi}, \quad \dot{\boldsymbol{v}}_a = \nabla \ddot{\chi} + (d/d_a)\{\nabla \times (\chi \hat{\boldsymbol{z}}) - \mathscr{G} \nabla \eta\}, \qquad (7a, b)$$

$$\dot{\boldsymbol{V}} = \nabla(\ddot{\chi} - \mathscr{G}\eta) + \nabla \times (\chi \hat{\boldsymbol{z}}), \qquad (7c)$$

where the depth-averaged velocity $\boldsymbol{V} = (d_e \boldsymbol{v}_e + d_a \boldsymbol{v}_a)/d$. To derive (7), we first express the dimensional pressure P_* in terms of χ_*. On combining (1a), (1c) and (1f), we find that $P_* + \rho d_a \ddot{\chi}_*/J_o$ is harmonic within $\mathscr{A}$, with zero normal derivative on the boundary $\mathscr{C}$ because both P_{*n} and χ_{*n} vanish there by (1g) and (1h). So,

$$P_* = -\rho d_a \ddot{\chi}_*/J_o + \mathrm{fn}(t), \qquad (8)$$

provided (1g) and (1h) hold, so the walls are impermeable and non-conducting. Equations (7a) and (7b) follow by substituting for P_* in (1a) and (1b).

By substituting the expression $\{\eta, \chi\} = \mathrm{e}^{\sigma t}\{H(x, y), K(x, y)\}$ into (5), we find that the growth rate σ satisfies

$$\tau H = \mathscr{G} \nabla^2 H, \quad H = \nabla^2 K \ \text{ within } \mathscr{A}; \qquad (9a, b)$$

$$\text{on } \mathscr{C}, \quad K_n = 0, \quad \mathscr{G} H_n = -K_s; \qquad (9c, d)$$

$$\tau = \sigma^2. \qquad (9e)$$

3. Edge wave propagating along a plane wall

To describe this solution, we take the x-axis along the wall, with $y = 0$ at the wall. By substituting $\{H, K\} = \{\tilde{H}(y), \tilde{K}(y)\}\mathrm{e}^{\mathrm{i}kx}$ into (9), we find that for $0 < y < \infty$,

$$\tilde{H}_{yy} - (k^2 + \tau/\mathscr{G})\tilde{H} = 0, \quad \tilde{K}_{yy} - k^2 \tilde{K} = \tilde{H}; \qquad (10a, b)$$

$$\text{on } y = 0, \quad \tilde{K}_y = 0, \quad \mathscr{G}\tilde{H}_y = -\mathrm{i}k\tilde{K}; \qquad (10c, d)$$

$$\text{as } y \to \infty, \quad \tilde{H} \to 0. \qquad (10e)$$

This determines the growth rate as functions of the real parameters k and $\mathscr{G}$. In (10e) we require $\tilde{H}$ to vanish at infinity; of course there are solutions periodic in both space and time which do not satisfy this condition (a stone tossed anywhere into the cell creates waves). In applying (10e), we choose to study disturbances which can be triggered near the wall, and which remain trapped there as they grow in time. We discuss the other, periodic solutions briefly at the end of this section.

The solution of (10a, b) satisfying (10c, e) is

$$\tilde{H} = \mathrm{e}^{-\ell y}, \quad \tilde{K} = (k\tau)^{-1}\mathscr{G}\{k\mathrm{e}^{-\ell y} - \ell \mathrm{e}^{-ky}\}; \qquad (11a, b)$$

$$\ell = k\sqrt{1 + \tau/(2\gamma)}, \quad \gamma = \mathscr{G}k^2/2. \qquad (11c, d)$$

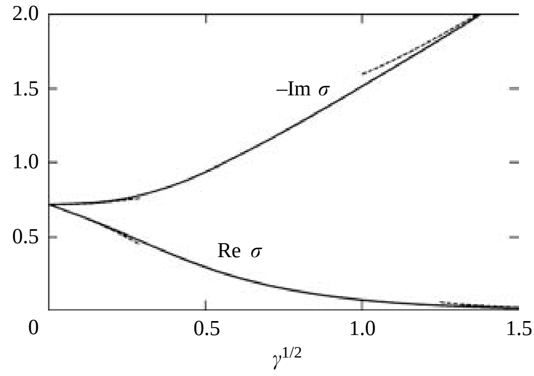

FIGURE 2. Dispersion relation (13) for edge waves. Broken curves, asymptotes: for $\gamma \to 0$, $\sigma \sim (-\mathrm{i})^{1/2} - \sqrt{\gamma/2} - \frac{3}{4}\mathrm{i}^{1/2}\gamma$; for $\gamma \to \infty$, $\sigma \sim -\mathrm{i}\sqrt{2\gamma}\{1 + \frac{1}{8}\gamma^{-2} + \frac{1}{8}\mathrm{i}\gamma^{-3}\}$.

Here we define $\sqrt{z}$ to be the square root with positive real part, i.e. we take the branch cut so that $-\pi < \arg z \leqslant \pi$. In (11$a$), we have taken $\tilde{H}(0) = 1$; this is consistent with our choice in (4d) of the amplitude a as the scale for η.

By applying the remaining condition (10d), we find that

$$(\tau + \mathrm{i})\sqrt{1 + \tau/(2\gamma)} = \mathrm{i}. \tag{12}$$

By squaring (12), we find that τ satisfies a quadratic equation, whose relevant root is

$$\tau = -\mathrm{i} - \gamma - \sqrt{\gamma^2 - 2\mathrm{i}\gamma}. \tag{13}$$

The redundant root is eliminated by showing that it does not satisfy (12) for small γ.

Figure 2 shows the growth rate σ calculated from (13) as a function of $\sqrt{\gamma}$. We see that these edge waves grow for all k, because $\mathrm{Re}\,\sigma > 0$; however, because $\mathrm{Re}\,\sigma$ decreases with increasing $\sqrt{\gamma}$, we conclude that when gravity is strong, it acts to stabilize the hydrostatic base state. We also see that because $\mathrm{Im}\,\sigma < 0$ for all k, the growing wave propagates only in the positive x-direction, i.e. in the unique direction $\hat{\boldsymbol{n}} \times \boldsymbol{B}_o$.

We now verify that that the wave grows by extracting power from the applied current via the component of disturbance current along the wall. By applying the energy identity (6) to a strip $\mathscr{A}$ of length $2\pi/k$ in the x-direction and extending from $y = 0$ to infinity, we find that

$$\dot{\mathscr{E}} = \int_0^{2\pi/k} (\dot{\eta}\,\chi_x)_{y=0}\,\mathrm{d}x = -\pi \mathrm{e}^{(\sigma+\bar{\sigma})t}\,\mathrm{Im}\{\bar{\sigma}\tilde{K}(0)\}, \tag{14a,b}$$

where $\bar{\sigma}$ is the complex conjugate of σ. (To derive (14b), we expressed the real variables $\dot{\eta}$ and χ_x in terms of the corresponding complex quantities. That gives $\dot{\eta}\chi_x$ as the sum of four terms: two of these integrate to zero over a wavelength; the other two can be expressed as $\mathrm{Im}\{\bar{\sigma}\tilde{H}(0)\tilde{K}(0)\}$. Equation (14$b$) follows since $\tilde{H}(0) = 1$.)

To prove that $\dot{\mathscr{E}} > 0$, we first use (11b,c) and (12) to show that $\tilde{K}(0) = \mathscr{G}/(\tau + \mathrm{i})$. Now, $\tau + \mathrm{i}$ lies in the second quadrant of the complex plane, because by (13) it is the difference of $-\gamma$, which lies on the negative real axis, and $\sqrt{\gamma^2 - 2\gamma\mathrm{i}}$ which lies in the fourth quadrant. So $\bar{\sigma}\tilde{K}(0)$ lies in the lower half-plane, because it is the quotient of $\bar{\sigma}$, which is in the first quadrant by figure 2, and $\tau + \mathrm{i}$ which is in the second quadrant. The right-hand side of (14*b*) is therefore positive, as required for instability.

Next, we discuss the inner-and-outer structure possessed by these edge waves in the weak gravity limit $\gamma \to 0$. Equation (13) then requires that $\tau \to -\mathrm{i}$, and by (11*c*), $\ell\sqrt{\gamma}/k \to (1-\mathrm{i})/2$. Consequently, in this limit ℓ is large, and (11*a*) shows that the interface displacement is confined to a thin wall layer of characteristic thickness $O(k^{-1}\sqrt{\gamma})$. By contrast, (11*b*) shows that the potential decays in y on the length scale k^{-1} set by the wavelength, i.e. slowly compared with the interface displacement. Moreover, by (11*a*)

$$\eta \sim \mathrm{e}^{\check{t}-\check{y}}\,\mathrm{e}^{\mathrm{i}(kx+\check{y}-\check{t})}, \tag{15}$$

for $\mathscr{G} \to 0$, where $\check{t} = t/\sqrt{2}$ and $\check{y} = y/\sqrt{2\mathscr{G}}$. It is easily verified that this expression satisfies the wave equation (5*a*) for vanishing $\mathscr{G}$. Physically, the expression describes a wave propagating away from the wall; the wave decays in $\check{y}$ but grows with time.

Lastly, we discuss the solutions of (10*a*–*d*) that are purely periodic in space and time. Lukyanov, El & Molokov (2001) use those solutions to describe reflection of a wave by the wall; they prescribe a non-zero amplitude at infinity, rather than requiring H to vanish there, and they find that the amplitude of the reflected wave exceeds that of the incident wave. This result leads them to propose that instability occurs by repeated wave reflection, but they do not show precisely how amplification on reflection would lead to exponential growth in time. Equation (15) provides a counter-example to their proposition because it cannot be interpreted in terms of incident and reflected waves.

4. Solution for an arbitrary finite domain in the limit $\mathscr{G} \to 0$

We now show that the separation of scales discussed above means that the edge-wave solution generalizes to arbitrary cell planforms. We define the outer limit as $\mathscr{G} \to 0$ with fixed x and y not on the wall. In that limit, (9*a*) simplifies to $H = 0$, so that the interface perturbation vanishes. Physically, this means that modes growing on the time scale T, which is independent of g, must be excited near the walls in a thin layer in which the outer scaling breaks down. As in the edge-wave solution, these waves decay as they leave the walls so that H vanishes in the core of the cell; consequently, $\nabla^2 K = 0$, by (9*b*). Because all time derivatives are lost in the outer limit, the growth rate must be determined by the coupling between the core flow and wall layers.

The outer problem cannot hold near the wall because the order of (9*a*) is lowered in the outer limit, making it impossible to satisfy both boundary conditions (9*c*) and (9*d*). We rescale to retain the highest derivatives within a boundary layer. First, continuity of current normal to the wall requires the change ΔK in potential across the wall layer of thickness δ to satisfy $\Delta K/\delta \sim K$; by (9*a*), $\tau \sim \mathscr{G}/\delta^2$; by (9*b*), $H \sim \Delta K/\delta^2$; and by (9*d*), $\mathscr{G}H/\delta \sim K$. By solving these equations for the unknowns ΔK, δ, K and τ, we find that $\tau \sim 1$, $\delta \sim \sqrt{\mathscr{G}}$, $\Delta K \sim H\mathscr{G}$ and $K \sim H\sqrt{\mathscr{G}}$. As in § 3, we choose $H \sim 1$; this is consistent with our choice in (4*d*) of the amplitude a as the scale for η.

We therefore define inner variables (with circumflexes) by

$$\hat{H} = H, \quad \hat{K} = \{K - K_w(s)\sqrt{\mathscr{G}}\}/\mathscr{G}, \quad \hat{s} = s, \quad \hat{n} = n/\sqrt{\mathscr{G}}, \tag{16a–d}$$

where $\hat{K}$ represents the change in potential across the wall layer, and $K_w(s)\sqrt{\mathscr{G}} = \lim_{n\to 0} K(s, n)$, i.e. the core potential evaluated at the wall.

In the inner limit defined by $\mathscr{G} \to 0$ with $\hat{n}$ fixed, (9) and (16) require

$$\tau \hat{H} = \hat{H}_{\hat{n}\hat{n}}, \quad \hat{H} = \hat{K}_{\hat{n}\hat{n}} \quad \text{for } 0 < \hat{n} < \infty; \tag{17a,b}$$

$$\text{at } \hat{n} = 0, \quad \hat{K}_{\hat{n}} = 0, \quad \hat{H}_{\hat{n}} = -K'_w(\hat{s}), \tag{17c,d}$$

and matching requires $\hat{H}$ and $\hat{K}_n$ to be finite at infinity. The prime denotes differentiation. Both boundary conditions (9c) and (9d) are applied in the inner problem (17).

The solution of (17) finite at infinity is

$$\hat{H} = \sigma^{-1} K'_w(\hat{s}) \mathrm{e}^{-\sigma \hat{n}}, \quad \hat{K}_{\hat{n}} = \tau^{-1} K'_w(\hat{s}) \{1 - \mathrm{e}^{-\sigma \hat{n}}\}, \tag{18a,b}$$

where τ is defined by (9c). We see that at the outer edge of the wall layer, the displacement vanishes, and the current is finite.

The matching condition on the solution is $\lim_{n\to 0} K_n = \lim_{\hat{n}\to\infty} \hat{K}_{\hat{n}}$, which can be written $\lim_{n\to 0} K_n = \tau^{-1} K'_w(s)$ by (18b). So within the core, the potential satisfies the oblique-derivative problem

$$\nabla^2 K = 0 \text{ within } \mathscr{A}; \tag{19a}$$

$$\text{on } \mathscr{C}, \quad \tau K_n = K_s. \tag{19b}$$

Boundary condition (19b) can be also be derived by eliminating $\hat{H}$ between the left-hand sides of (17a) and (17b), then integrating the resulting expression across the wall layer, and finally applying (17c) and (17d). That argument identifies (19b) as a jump condition that is independent of the internal structure of the wall layer.

The growth rate σ is thus determined by the core problem (19), but the boundary condition (19b) on that problem is imposed by the wall layers. Gravity is essential to the existence of the edge waves that set up the jump condition, but that condition is asymptotically independent of $\mathscr{G}$, and so of gravity.

By inspection, (19) is invariant under conformal mapping so that τ is independent of the planform. But for a half-space, (13) implies that for $\mathscr{G} \to 0$, $\tau \to -\mathrm{i}$. Consequently

$$\lim_{\mathscr{G}\to 0} \sigma = \mathrm{e}^{-\mathrm{i}\pi/4}, \tag{20}$$

for all planforms.

5. Channel waves

This is the simplest case having a finite critical value $\mathscr{G}_c$. We verify the energy identity (6) by showing that the right-hand side of (6b) vanishes for $\mathscr{G} > \mathscr{G}_c$, but is positive otherwise; the disturbance therefore oscillates neutrally or grows according to the power supplied to it at the boundary. We also verify (20), by showing that it is satisfied by the limiting growth rate for a channel.

As in § 3, we let $\{H, K\} = \{\tilde{H}(y), \tilde{K}(y)\}\mathrm{e}^{\mathrm{i}kx}$ To find the eigenfunctions, we note that by figure 6(a) of Davidson & Lindsay, the base state loses stability to waves with $k = 0^+$. So, we let

$$\tilde{H} = \tilde{H}_1 + O(k), \quad \tilde{K} = k^{-1}\tilde{K}_0 + \tilde{K}_1 + O(k), \quad \tau = \tau_1 + O(k), \tag{21a–d}$$

where the coefficients $\tilde{H}_1, \tilde{K}_0, \tau_1, \cdots$ are independent of k. In (21c), $\tilde{K}$ varies as k^{-1} at leading order because as $k \to 0$, the current χ_x must remain non-zero for instability.

The boundary-value problem for $\tilde{H}$ and $\tilde{K}$ is identical with (10), except that (10c,d) are now applied at both walls at $y = \pm 1$. By substituting (21) into the modified form

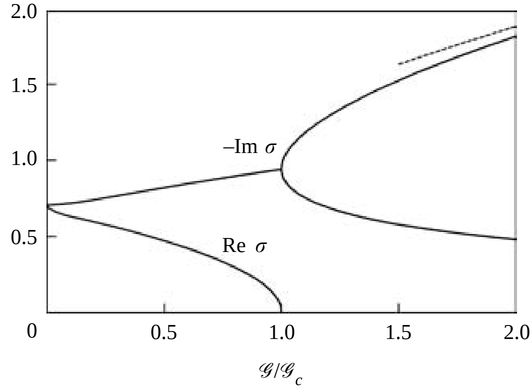

FIGURE 3. Dispersion relation (24) for channel waves. Broken curve, dispersion relation $\mathrm{Im}\,\sigma = \frac{1}{2}\pi\sqrt{\mathscr{G}}$ for the lowest allowable pure gravity wave. Critical value, $\mathscr{G}_c = 0.7326$.

of (10), we find that $\tilde{K}_0$, $\tilde{K}_1$ and $\tilde{H}_1$ satisfy

$$\tilde{K}_0'' = 0, \quad \tilde{H}_1'' - \ell^2 \tilde{H}_1 = 0, \quad \tilde{K}_1'' = \tilde{H}_1 \quad \text{for } |y| < 1; \tag{22a–c}$$

$$\text{on } |y| = 1, \quad \tilde{K}_0' = 0, \quad \tilde{K}_1' = 0, \quad \mathscr{G}\tilde{H}_1' = -\mathrm{i}\tilde{K}_0; \tag{22d–f}$$

$$\ell^2 = \tau_1/\mathscr{G}. \tag{22g}$$

By solving (22), we find that

$$\lim_{k\to 0} \tilde{H}(y) = (\sinh \ell y)/\sinh \ell; \tag{23a}$$

$$\lim_{k\to 0}\{\tilde{K}(y) - \mathrm{i}\ell\mathscr{G}(\coth \ell)/k\} = c_0 + \{\sinh \ell y - \ell y \cosh \ell\}/(\ell^2 \sinh \ell). \tag{23b}$$

We have normalized the solution so that $\tilde{H}(1) = 1$. (The integration constant c_0 is not needed here, but can be found by continuing to $O(k)$ in the expansion.)

The growth rate is found either by proceeding to $O(k)$, or by setting $k = 0$ in the full dispersion relation $\ell^2\{1 + \mathscr{G}^2(\ell^2 - k^2)^2\} + k^2 = k\ell\,\{\tanh \ell \,\coth k + \tanh k \,\coth \ell\}$, i.e. Davidson & Lindsay's (5.1), simplified without approximation. By either method,

$$\mathscr{G}^2 \ell^5 + \ell = \tanh \ell, \tag{24}$$

where we have used (22g).

Figure 3 shows σ calculated as a function of $\mathscr{G}/\mathscr{G}_c$ from (24) and (22g), i.e. $\sigma^2 = \mathscr{G}\ell^2$. We see that as $\mathscr{G} \to 0$, $\mathrm{Re}\,\sigma$ and $-\mathrm{Im}\,\sigma$ for a channel approach the limit (20) predicted by the asymptotic analysis for a cell of arbitrary planform. Because σ approaches a limit as $\mathscr{G} \to 0$, the complex wavenumber $\ell \to \infty$, so that (23a) requires that $\hat{H} \to 0$ outside thin boundary layers.

Lastly, the disturbance is pumped at the wall if $\mathscr{G} < \mathscr{G}_c$, but not otherwise. The claim follows by using the eigenfunctions (23) to show that

$$\frac{k}{2\pi} \oint \dot{\eta}\chi_s \,\mathrm{d}s = \mathscr{G}^{3/2}|\ell|^2 \mathrm{e}^{2\sigma_r t} \frac{\sinh 2\ell_r}{\cosh 2\ell_r - \cos 2\ell_i}, \tag{25}$$

where $\ell_r = \mathrm{Re}\,\ell$, and $\ell_i = \mathrm{Im}\,\ell$. Power is therefore fed to the disturbance if $\ell_r > 0$; then, $\mathrm{Re}\,\sigma > 0$ because $\sigma = \ell\sqrt{\mathscr{G}}$.

6. Discussion

The growth rate for $\mathscr{G} \to 0$ can also be found for an arbitrary domain without using conformal mapping. We first prove that the depth-averaged velocity $\boldsymbol{V}=0$ within the core. At the outer edge of the wall layer, $\eta \to 0$ by (18), and by (7*c*), $\dot{\boldsymbol{V}} = \nabla \ddot{\chi} + \nabla \times (\chi \hat{\boldsymbol{z}})$. Consequently $\hat{\boldsymbol{n}} \cdot \dot{\boldsymbol{V}} = \ddot{\chi}_n - \chi_s$, which vanishes by (19*b*). But within the core $\nabla \cdot \dot{\boldsymbol{V}} = 0$; also $\nabla \times \dot{\boldsymbol{V}} = 0$ by (7*c*) and the core equation $\nabla^2 K = 0$. Because $\dot{\boldsymbol{V}}$ is irrotational and solenoidal throughout the core, and $\hat{\boldsymbol{n}} \cdot \dot{\boldsymbol{V}} = 0$ on the core boundary, $\dot{\boldsymbol{V}} = 0$.

Next, by eliminating χ between the momentum equations (7*a*) and (7*b*), then using the identity $\nabla \dot{\eta} = \nabla^2 \boldsymbol{v}_e$ following from (5*b*) and (7*a*), we find that

$$\ddot{\boldsymbol{v}}_a - \ddot{\boldsymbol{v}}_e = (d/d_a)\{\boldsymbol{v}_e \times \hat{\boldsymbol{z}} - \mathscr{G}\nabla^2 \boldsymbol{v}_e\}. \tag{26}$$

But within the core, the term $\mathscr{G}\nabla^2 \boldsymbol{v}_e$ is negligible for vanishing $\mathscr{G}$; moreover, the depth-averaged velocity $\boldsymbol{V}$ vanishes identically there, so that $\boldsymbol{v}_a = -\boldsymbol{v}_e d_e/d_a$, and by (26)

$$\ddot{\boldsymbol{v}}_e = \hat{\boldsymbol{z}} \times \boldsymbol{v}_e. \tag{27}$$

This holds pointwise within the core in the limit $\mathscr{G} \to 0$. (Boundary layer mechanics are built into (27), because the jump condition is used to prove that $\boldsymbol{V}=0$ in the core.) Because (27) is an ODE in $\boldsymbol{v}_e$, the growth rate is identical for all planforms, as we also found by another argument in § 4.

Lastly, we compare our calculation of the growth rate with a momentum integral analysis given by Davidson & Lindsay (1998, § 5.2). To clarify the relation between the two approaches, we give a new derivation of their result; neither uses the jump condition. Because $\boldsymbol{V}$ is solenoidal with vanishing normal component on the cell wall, $\int_{\mathscr{A}} \boldsymbol{V}\, \mathrm{d}\mathscr{A} = 0$, by the identity $V_i = \partial(x_i V_j)/\partial x_j$ and the divergence theorem. That relation allows the area integral of $\boldsymbol{v}_a$ to be written in terms of $\boldsymbol{v}_e$. By integrating (26) over $\mathscr{A}$, and eliminating $\boldsymbol{v}_a$, we find that for $\mathscr{G} \to 0$

$$\ddot{\boldsymbol{M}} = \hat{\boldsymbol{z}} \times \boldsymbol{M}, \quad \boldsymbol{M} = \int_{\mathscr{A}} \boldsymbol{v}_e\, \mathrm{d}\mathscr{A}. \tag{28a, b}$$

This is equivalent to Davidson & Lindsay's momentum integral. In (28*a*) we have taken as negligible for small $\mathscr{G}$ a term $\mathscr{G} \int \nabla^2 \boldsymbol{v}_e\, \mathrm{d}\mathscr{A}$. Although (28) does not contain $\mathscr{G}$, it does not follow that the instability mechanism is independent of g, because in forming (28), all spatial information is lost. That is obvious in the light of our asymptotic analysis in § 4, because the wall layer affects the core flow only through a jump condition that is independent of $\mathscr{G}$. Our analysis shows that whenever the shallow-water theory holds, gravity is essential to the instability because magnetogravity waves trapped in a thin wall layer impose the correlation needed for power to be fed from the mean state to the disturbance. The pendulum model does not describe that physics, because for it the limit of vanishing g is not singular.

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